1901001101040001
EXAMINATION FEBRUARY-MARCH 2024
MASTER OF ARTS (PART - I) (EXTERNAL)
MATHEMATICS – LEVEL 4
ORDINARY DIFFERENTIAL EQUATIONS

[**Time: As Per Schedule]** **[Max. Marks: 100]**

Instructions:
1. Fill up strictly the following details on your answer book
   a. Name of the Examination : **MASTER OF ARTS (PART - I) (EXTERNAL)**
   b. Name of the Subject : **MATHEMATICS – LEVEL 4 ORDINARY DIFFERENTIAL EQUATIONS**
   c. Subject Code No : **1901001101040001**
2. Sketch neat and labelled diagram wherever necessary.
3. Figures to the right indicate full marks of the question.
4. All questions are compulsory.
5. There are five questions in this questions paper.

Seat No:

Student's Signature

Q.1 (I) Let $\Phi(t)$be a fundamental matrix of $x'(t) = A(t)x(t)$ with $\Phi(t_0) = I$. Then prove that $x'(t) = A(t)x(t)$ is strongly stable if and only if there exists a positive constant M such that $\|\Phi(t)\| \leq M, \|\Phi^{-1}(t)\| \leq M$ for $t \geq t_0$ **7**

(II) A system $x'(t) = A(t)x(t)$ is restrictively stableiff it is reducible to zero. **7**

(III) Show that the second order differential equation $u'' + \frac{2u'}{t+1} = 0$ is stable but not uniformly stable. **6**

OR

(I) Let there exists a positive constant M such that $\|\Phi(t)\Phi^{-1}(s)\| \leq M$, for $t_0 \leq s < t < \infty$ and let $f(t,x)$ satisfies the inequality $\|f(t,x)\| \leq \gamma(t)\|x\|$ where $\gamma(t)$ is non-negative continuous function such that $\int_{t_0}^{\infty} \gamma(t)dt < \infty$. Then there exists a positive constant L such that if $t \geq t_0$ every solution **7**

$x(t)$ *of* $x' = A(t)x + f(t,x)$ for which $\|x(t_1)\| \leq C/M$ is defined $\forall t_1 \geq t_0$ and satisfies $\|x(t)\| \leq M_1 e^{-\propto(t-t_1)}\|x(t_1)\|, \forall t \geq t_1$ and for some $M_1 > 0$.

(II) If $x'(t) = A(t)x(t)$ is uniformly (or restrictively) stable and $\int_{t_0}^{\infty}\|B(t)\|dt < \infty$ then the system $x' = (A(t) + B(t))x$ is also uniformly (or restrictively) stable. **7**

(III) For the differential equation $u' = -u(1-u)$ show that (a) the solution $u(t) = 0$ is asymptotically stable. (b) the solution u(t) = 1 is unstable. **6**

Q.2 (I) Let $u_1(t)$ *and* $u_2(t)$ be two linearly independent solutions of $u'' + a(t)u = 0$ on the interval $0 \leq t < \infty$. Then the general solution u(t) of the inhomogeneous equation $u' + a(t)u = g(t)$ is given by **7**

$u(t) = c_1u_1(t) + c_2u_2(t) + \int_0^t [u_1(s)u_2(t) - u_1(t)u_2(s)]g(s)ds$. Where c_1, c_2 are arbitrary constant.

(II) Show that the critical point (0,0) of the Vander-Pol's equation $u'' + \varepsilon(u^2 - 1)u' + u = 0$, where ε is a positive constant, is always unstable **7**

(III) If $u = \phi(t)$ is a solution of the Riccati equation $u' = p(t)u + q(t)u^2 + r(t)$, show that this equation has other solution of the form $u = \phi(t) = -\frac{1}{\psi(t)}$ where $\psi(t)$ is a solution of an equation of the form $u' = a(t)u + b(t)$. **6**

OR

(I) If u and u'' belongs to $L^2[0,\infty)$ then u' also belongs to $L^2[0,\infty)$. **7**

(II) Let b(t) be the continuously differentiable on [0, ∞). If b(t) → 0 as t→ ∞ and $\int_0^{\infty}|b'(s)|\, ds < \infty$, then all the solutions of $u'' + (1 + b(t))u = 0$ are bounded over $[0,\infty)$ **7**

(III) If $\|u\|$ *and* $\|u''\|$ are bounded then $\|u'\|$ is also bounded. **6**

Q.3 (I) If $g(t,u)$ is a continuous function of t and u in a closed bounded region R(a, b) and satisfies the Lipschitz condition in R then there exists a unique solution u(t) to the initial value problem $u' = g(t,u), u(t_0) = u_0$ on the interval $J: |t - t_0| \leq h$, where $h = \min(a, \frac{b}{M})$and $|g(t,u(t))| \leq M$ on R(a,b). **7**

(II) If $\emptyset_1(t), \emptyset_2(t), \ldots \ldots, \emptyset_n(t); r_1 < t < r_2$ is a set of Linearly independent solutions of $x'(t) = A(t)x(t)$. Then the linear combination $\sum_{j=1}^{n} c_j \emptyset_j(t)$ never vanishes on $r_1 < t < r_2$ unless $c_1 = c_2 = \cdots .. = c_n = 0$ **7**

(III) Determine whether characteristic polynomial $L(\lambda) = \lambda^3 + 5\lambda^2 + 9\lambda + 5$ is stable or not. **6**

OR

(I) Let a(t) and b(t) be continuous on the interval I and let $t_0 \in I$ then there exists a unique solution u(t) to the initial value problem $u' = a(t)u + b(t);\ u(t_0) = u_0$ on I. **7**

(II) Let u and v be a non-negative continuous function on some interval$t_0 \leq t \leq t_0 + \alpha$. Also let function f(t) be positive continuous and monotonically non-decreasing on $t_0 \leq t \leq t_0 + \alpha$, and satisfy the inequality $u(t) \leq f(t) + \int_{t_0}^{t} u(s)v(s)ds;\ \ t \in [t_0, t_0 + \alpha]$ then the inequality $u(t) \leq f(t) \exp[\int_{t_0}^{t} v(s)ds\ ; t \in [t_0, t_0 + \alpha]$ holds. **7**

(III) If g(t, u) is a continuous function in a strip $S: t_0 \leq t \leq t_0 + a, |u| < \infty$ and g is monotonically non-increasing in u for each fixed t in S then the initial value problem $u' = g(t,u), u(t_0) = u_0$ has a unique solution u(t) defined on the interval $[t_0, t_0 + a]$. **6**

Q.4 (I) Let u and v be non-negative continuous functions on $t_0 \leq t \leq t_0 + a$ satisfying $u(t) \leq c + \int_{t_0}^{t} u(s)v(s)ds,\ \ t \in [t_0, t_0 + a]$, where c is a non-negative constant, then the inequality $u(t) \leq c\exp\left(\int_{t_0}^{t} v(s)ds\right),\ \ t \in [t_0, t_0 + a]$ holds. **7**

(II) If K_1 *and* K_2 are +ve constants and u is a non-negative continuous function on the interval $\alpha \leq t \leq \beta$ satisfying the inequality, $u(t) \leq K_1 + K_2 \int_{\alpha}^{t} u(s)ds$ show that $u(t) \leq K_1 \exp(K_2(t-\alpha))$ **7**

(III) Apply Picard's method to initial value problem $u' = 2tu + 4t, u(0) = 1$ and show that the successive approximations tends to the exact solution. **6**

OR

(I) A fundamental system of solutions of $x'(t) = A(t)x(t), x(t_0) = x_0$ exists. **7**

(II) Prove that a complex number λ is a characteristic exponent of $x'(t) = A(t)x(t)$. Where $A(t+w) = A(t)$ if and only if there exists a non-trivial solution of this system in the form $e^{\lambda t}p(t)$ where $p(t+w) = p(t)$. **7**

(III) Discuss the solutions of Mathieu's equation $u'' + (\delta^2 + \varepsilon \cos 2t)u = 0$ with initial conditions $u_1(0) = 1, u_1'(0) = 0, u_2(0) = 0, u_2'(0) = 1.$ **6**

Q.5 (I) Let P(t) be a non constant periodic solution of $x' = f(x)$ and let f(x) be continuously differentiable at all points of the trajectory. Further suppose zero is a simple characteristic exponent of $z' = A(t)z$ and every other characteristic exponent of this variational system has a negative real part, then P(t) is orbitally asymptotically stable. **7**

(II) A fundamental matrix $\Phi(t)$ is a solution of $\Phi'(t) = A(t)\Phi(t);\ \Phi(t_0) = I$. Further the solution $x(t)$ *of* $x'(t) = A(t)x(t)$ satisfying $x(t_0) = x_0$ can be written as $x(t) = \Phi(t)x_0$. **7**

(III) Draw phase portrait and determine type of critical point of the following systems. **6**

(i) $x_1' = 3x_1 + 2x_2$, $x_2' = 4x_1 - x_2$

(ii) $x_1' = x_1 + 3x_2$, $x_2' = -6x_1 + 5x_2$

OR

(I) Let $\Phi(t)$ be a fundamental matrix of $x'(t) = A(t)x(t)$ with $\Phi(t_0) = I$. Then prove that $x'(t) = A(t)x(t)$ is uniformly asymptotically stable if and only if there exists a positive constant M and $\propto$ such that $\|\emptyset(t)\emptyset^{-1}(s)\| \leq Me^{-\propto(t-s)}$ for all $t_0 \leq s \leq t < \infty$. **7**

(II) Let P(t) be a non constant periodic solution of $x' = f(x)$ and let f(x) be continuously differentiable at all points of the trajectory. Further suppose zero is a simple characteristic exponent of $z' = A(t)z$ and every other characteristic exponent of this variational system has a negative real part, then P(t) is orbitally asymptotically stable. **7**

(III) Every normal solution of $u'' + P(t)u = 0$ is bounded as $t \to \infty$ and hence stable iff the characteristic exponent of u" + P(t)u = 0 are purely imaginary. **6**

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